

# AN ANSATZ FOR HYPERKÄHLER 8-MANIFOLDS WITH TWO COMMUTING ROTATING KILLING FIELDS

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**ABSTRACT.** We consider a hyperkähler 8-manifold admitting either a  $U(1) \times \mathbb{R}$ , or a  $U(1) \times U(1)$  action, where the first factor preserves  $g$  and  $I$ , and acts on  $\omega_2 + i\omega_3$  by multiplying it by itself, while the second factor preserves  $g$  and acts triholomorphically. Such data can be reduced to a single function  $H$  of two complex variables and two real variables satisfying 6 equations of Monge-Ampère type, which can be compactly written down using a Poisson bracket.

## 1. INTRODUCTION

In recent years, both mathematicians (for example [5], [6] and [1]) and physicists (for example [3]) became interested in hyperkähler manifolds with so-called “rotating” Killing fields: these are Killing fields whose flow preserves a complex structure in the  $S^2$  of complex structures of a hyperkähler metric, say  $I$ , and rotates the other two,  $J$  and  $K$ , in the plane spanned by  $J$  and  $K$ . In dimension 4, those already had been studied by authors such as Boyer and Finley in [2], and their symmetry reduction leads to the Boyer-Finley-LeBrun equation (see [2] and the later work by LeBrun in [7]).

Consider flat quaternionic space in real dimension 8. This can be described as  $\mathbb{C}^4$  with coordinates  $(q^1, q^2, p_1, p_2)$ , together with the Kähler form

$$\omega_1 = \frac{i}{2} \left( \sum_{j=1}^2 dq^j \wedge d\bar{q}^j + \sum_{j=1}^2 dp_j \wedge d\bar{p}_j \right)$$

as well as the holomorphic symplectic form

$$\omega_+ = \sum_{j=1}^2 dq^j \wedge dp_j$$

The function

$$\Omega = \sum_{j=1}^2 |q^j|^2 + \sum_{j=1}^2 |p_j|^2$$

is a Kähler potential for  $\omega_1$ ; in other words

$$\omega_1 = \frac{i}{2} \partial \bar{\partial} \Omega$$

(note the unconventional factor of  $1/2$ ). Let  $t$  and  $c$  be real variables, so that  $(e^{it}, c) \in \mathrm{U}(1) \times \mathbb{R}$ . The action  $L$  of  $\mathrm{U}(1) \times \mathbb{R}$  on  $\mathbb{C}^4$  defined by

$$L_{(e^{it}, c)}(q^1, q^2, p_1, p_2)^T = (q^1 + c, q^2, e^{it}p_1, e^{it}p_2)^T$$

preserves the complex structure of  $\mathbb{C}^4$  which we denote by  $I$ , as well as  $\omega_1$ , and its effect on  $\omega_+$  is as follows:

$$L_{(e^{it}, c)}^*(\omega_+) = e^{it}\omega_+$$

We let

$$(1) \quad \begin{aligned} u &= q^1 + \bar{q}^1 \\ v &= i(\bar{q}^1 - q^1) \\ q &= q^2 \\ \zeta &= p_2/p_1 \\ \rho &= \ln(|p_1|^2) \\ \theta &= i(\ln(\bar{p}_1) - \ln(p_1)) \end{aligned}$$

We note that, in the new coordinates  $(u, v, q, \rho, \theta, \zeta)$ ,  $\Omega$  is independent of  $\theta$ . It is not independent of  $u$  though. But  $\Omega$  can be replaced by  $\Omega + F + \bar{F}$ , where  $F$  is a holomorphic function on  $\mathbb{C}^4$ . We replace  $\Omega$  by the following

$$\Omega' = \Omega - \frac{1}{2}(q^1)^2 - \frac{1}{2}(\bar{q}^1)^2$$

When expressed in the new coordinates,  $\Omega'$  becomes independent of both  $\theta$  and  $u$ , and is equal to the following function

$$H(q, \zeta, v, \rho) = \frac{1}{2}v^2 + |q|^2 + e^\rho(1 + |\zeta|^2)$$

We introduce the following bracket for pairs of functions of  $(q, \zeta, v, \rho)$ , defined by the following bivector

$$(2) \quad \{-, -\} = e^{-\rho}(i\partial_v \wedge \partial_\rho + i\zeta\partial_\zeta \wedge \partial_v + \partial_\zeta \wedge \partial_q)$$

It is easy to check that the Schouten-Nijenhuis bracket of this bivector with itself vanishes; in other words, the bracket is Poisson. Then it is

straightforward to check that the following equations hold:

$$(3) \quad \{H_\rho, iH_v\} = 1$$

$$(4) \quad \{H_{\bar{\zeta}}, H_{\bar{q}}\} = 1$$

$$(5) \quad \{H_\rho, H_{\bar{q}}\} = \bar{\zeta}$$

$$(6) \quad \{iH_v, H_{\bar{\zeta}}\} = 0$$

$$(7) \quad \{H_\rho, H_{\bar{\zeta}}\} = 0$$

$$(8) \quad \{iH_v, H_{\bar{q}}\} = 0$$

We claim that this is, in a sense, the general case for such types of actions. More precisely, we have

**Theorem 1.1.** *Let  $H(q, \zeta, v, \rho)$  be a real-valued function on  $\mathbb{C}^2 \times \mathbb{R}^2$  which satisfies equations (3)-(8). Then, on  $\mathbb{C}^2 \times \mathbb{R}^2 \times \mathrm{U}(1) \times \mathbb{R}$ , with coordinates  $(q, \zeta, v, \rho, \lambda, u)$ , we have the natural projection*

$$\pi : \mathbb{C}^2 \times \mathbb{R}^2 \times \mathrm{U}(1) \times \mathbb{R} \rightarrow \mathbb{C}^2 \times \mathbb{R}^2$$

*mapping  $(q, \zeta, v, \rho, \lambda, u)$  to  $(q, \zeta, v, \rho)$ . We introduce new coordinates on  $\mathbb{C}^2 \times \mathbb{R}^2 \times \mathrm{U}(1) \times \mathbb{R} \simeq \mathbb{C}^4$ :*

$$q^1 = \frac{1}{2}(u + iv)$$

$$q^2 = q$$

$$p_1 = e^{\rho/2}\lambda$$

$$p_2 = e^{\rho/2}\lambda\zeta$$

*Then, if we let  $\Omega(q^1, q^2, p_1, p_2)$  be  $\pi^*(H)$  expressed in the new coordinates  $(q^i, p_i)$ , and if we let*

$$\omega_1 = \frac{i}{2}\partial\bar{\partial}\Omega$$

$$\omega_+ = \sum_{j=1}^2 dq^j \wedge dp_j$$

*then  $\mathbb{C}^4$  with its complex structure  $I$ , together with  $\omega_1$  and  $\omega_+$  is hyperkähler having a  $\mathrm{U}(1) \times \mathbb{R}$  action  $L$*

$$L_{(e^{it}, c)}(q^1, q^2, p_1, p_2)^T = (q^1 + c, q^2, e^{it}p_1, e^{it}p_2)^T$$

*which preserves  $I$  and  $\omega_1$ , and such that*

$$(9) \quad L_{(e^{it}, c)}^*(\omega_+) = e^{it}\omega_+$$

*Conversely, any hyperkähler 8-dimensional manifold having a free  $\mathrm{U}(1) \times \mathbb{R}$  action preserving  $I$  and  $\omega_1$  and acting on  $\omega_+$  as in (9) can be locally*

described by a Kähler potential function with respect to  $I$  which is a real-valued function of (an open subset of)  $\mathbb{C}^2 \times \mathbb{R}^2$  satisfying equations (3)-(8).

*Proof.* We start by proving the converse. It can be shown that there exist holomorphic coordinates  $(q^1, q^2, p_1, p_2)$  with respect to  $I$  such that

$$\omega_+ = \sum_{j=1}^2 dq^j \wedge dp_j$$

and the action  $L$  takes the required form

$$L_{(e^{it}, c)}(q^1, q^2, p_1, p_2)^T = (q^1 + c, q^2, e^{it}p_1, e^{it}p_2)^T$$

This is in a way an equivariant version of the celebrated Darboux theorem in symplectic geometry. We introduce the coordinates

$$\begin{aligned} u &= q^1 + \bar{q}^1 \\ v &= i(\bar{q}^1 - q^1) \\ q &= q^2 \\ r &= |p_1| \\ \theta &= \frac{i}{2} \ln(\bar{p}_1) - \frac{i}{2} \ln(p_1) \\ \zeta &= \frac{p_2}{p_1} \end{aligned}$$

The inverse coordinate transformations are given by

$$\begin{aligned} q^1 &= \frac{1}{2}(u + iv) \\ q^2 &= q \\ p_1 &= re^{i\theta} \\ p_2 &= re^{i\theta}\zeta \end{aligned}$$

The coordinate vector fields in the two coordinate systems are related by

$$\begin{aligned} \partial_{q^1} &= \partial_u - i\partial_v \\ \partial_{q^2} &= \partial_q \\ \partial_{p_1} &= e^{-i\theta} \left( -\frac{\zeta}{r} \partial_\zeta + \frac{1}{2} \partial_r - \frac{i}{2r} \partial_\theta \right) \\ \partial_{p_2} &= \frac{1}{r} e^{-i\theta} \partial_\zeta \end{aligned}$$

We introduce the following Poisson bracket for pairs of functions on  $\mathbb{C}^4$ :

$$\begin{aligned} & \{-, -\}_{\text{original}} \\ &= \sum_{j=1}^2 \partial_{p_j} \wedge \partial_{q^j} \\ &= e^{-i\theta} \left( \left( -\frac{\zeta}{r} \partial_\zeta + \frac{1}{2} \partial_r - \frac{i}{2r} \partial_\theta \right) \wedge (\partial_u - i\partial_v) + \frac{1}{r} \partial_\zeta \wedge \partial_q \right) \end{aligned}$$

If  $\Omega$  is a Kähler potential for  $I$ , then the equations that  $\Omega$  must satisfy for the metric to be hyperkähler are the following symplectic Monge-Ampere equations (see [4]):

$$(10) \quad \{\Omega_{\bar{p}_i}, \Omega_{\bar{q}^j}\}_{\text{original}} = \delta_j^i$$

$$(11) \quad \{\Omega_{\bar{q}^i}, \Omega_{\bar{q}^j}\}_{\text{original}} = 0$$

$$(12) \quad \{\Omega_{\bar{p}_i}, \Omega_{\bar{p}_j}\}_{\text{original}} = 0$$

The equations (3)-(8) are the symmetry reductions of these equations. The Kähler potential  $\Omega$  can be replaced with

$$\Omega' = \Omega + F + \bar{F}$$

where  $F$  is a holomorphic function of the coordinates  $q^1, q^2, p_1$  and  $p_2$ . We claim that there is a holomorphic function  $F$  such that in the  $(u, v, q, r, \theta, \zeta)$  coordinates,  $\Omega'$  does not depend on  $u$  nor  $\theta$ . This follows from the fact  $\partial_u$  and  $\partial_\theta$  commute, and are each real parts of holomorphic vector fields. Denote by  $K$  the function of the coordinates  $(q, \zeta, v, r) \in \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}_+$  whose pullback to  $\mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times (\mathbb{R}/(2\pi\mathbb{Z}))$  with coordinates  $(q, \zeta, v, r, u, \theta)$  by the natural projection is equal to  $\Omega'$ .

We denote by  $\{-, -\}$  the Poisson bracket defined by (2). Then equation (11) yields equation (8), and equation (12) yields equation (7). The first claim is straightforward. Let us prove the second claim. Equation (12) implies:

$$\begin{aligned} & \left\{ -\frac{\bar{\zeta}}{r} K_{\bar{\zeta}} + \frac{1}{2} K_r, \frac{1}{r} K_{\bar{\zeta}} \right\} \\ &= \frac{i}{2r^2} \left( -\frac{\bar{\zeta}}{r} K_{\bar{\zeta}} + \frac{1}{2} K_r \right) K_{v\bar{\zeta}} - \frac{i}{2r^2} \left( -\frac{\bar{\zeta}}{r} K_{v\bar{\zeta}} + \frac{1}{2} K_{vr} \right) K_{\bar{\zeta}} \\ &= \frac{i}{4r^2} (K_r K_{v\bar{\zeta}} - K_{\bar{\zeta}} K_{vr}) \end{aligned}$$

But we also have that

$$\begin{aligned} & \left\{ -\frac{\bar{\zeta}}{r} K_{\bar{\zeta}} + \frac{1}{2} K_r, \frac{1}{r} K_{\bar{\zeta}} \right\} \\ &= \frac{1}{2} \left\{ K_r, \frac{1}{r} K_{\bar{\zeta}} \right\} \\ &= -\frac{i}{4r^2} K_{rv} K_{\bar{\zeta}} + \frac{i}{4r^2} K_r K_{v\bar{\zeta}} + \frac{1}{2r^2} \{r K_r, K_{\bar{\zeta}}\} \end{aligned}$$

And then equation (7) follows, using

$$(13) \quad \rho = \ln(r^2)$$

Using similar calculations, one can show that the remaining equations, namely (3)-(6), follow using the system (10). We provide a few details. System (10) implies

$$\begin{aligned} \left\{ -\frac{\bar{\zeta}}{r} K_{\bar{\zeta}} + \frac{1}{2} K_r, K_{\bar{q}^j} \right\} &= \delta_j^1 + \frac{i}{2r} \left( -\frac{\bar{\zeta}}{r} K_{\bar{\zeta}} + \frac{1}{2} K_r \right) K_{v\bar{q}^j} \\ \left\{ \frac{1}{r} K_{\bar{\zeta}}, K_{\bar{q}^j} \right\} &= \delta_j^2 + \frac{i}{2r^2} K_{\bar{\zeta}} K_{v\bar{q}^j} \end{aligned}$$

Equivalently, one can use the second equation to replace the first equation by a simpler one. One then gets the following simpler system:

$$\begin{aligned} \frac{1}{2} \{K_r, K_{\bar{q}^j}\} &= \delta_j^1 + \bar{\zeta} \delta_j^2 + \frac{i}{4r} K_r K_{v\bar{q}^j} \\ \left\{ \frac{1}{r} K_{\bar{\zeta}}, K_{\bar{q}^j} \right\} &= \delta_j^2 + \frac{i}{2r^2} K_{\bar{\zeta}} K_{v\bar{q}^j} \end{aligned}$$

The first equation above, with  $j = 1$ , yields equation (3), and with  $j = 2$ , yields equation (5). The second equation above yields equations (6) ( $j = 1$ ) and (4) ( $j = 2$ ).

The other direction of the proof is easier, and can be obtained by going “backwards” in the argument above.  $\square$

*Remark 1.2.* Since the proof above depends only on the generating vector fields of the  $U(1) \times \mathbb{R}$  action, one may just as well consider a  $U(1) \times U(1)$  action instead, where the first  $U(1)$  factor preserves  $g$  and  $I$  and rotates  $\omega_2$  and  $\omega_3$ , and the second factor preserves  $g$  and is triholomorphic. A similar result holds in this case. We present a non-trivial example of the latter type in the following section.

*Remark 1.3.* We remark that, on  $\mathbb{C}^2 \times \mathbb{C}^2$ , for functions that are pull-backs of functions on  $\mathbb{C}^2 \times \mathbb{R}^2$ , or in other words, for functions that are

independent of  $\theta$  and  $u$ , the two Poisson brackets are related by

$$\{-, -\} = \frac{1}{\bar{p}_1} \{-, -\}_{\text{original}}$$

(mod  $\partial_\theta$  and  $\partial_u$ ). This can be used to provide another quick proof for why  $\{-, -\}$  is Poisson, using the fact that  $\{-, -\}_{\text{original}}$  is a holomorphic Poisson bracket, and that  $\bar{p}_1$  is antiholomorphic.

## 2. EXAMPLE: THE CALABI METRIC ON $T^*(\mathbb{C}P^2)$

Using a local affine chart  $(z^1, z^2)$  on an affine subset of  $\mathbb{C}P^2$ , and corresponding fibre coordinates  $(w_1, w_2)$  on the cotangent bundle restricted to that affine subset, then the coordinates  $(z^1, z^2, w_1, w_2)$  are local holomorphic Darboux coordinates for the natural holomorphic symplectic structure on the cotangent bundle  $T^*(\mathbb{C}P^2)$ . The Calabi hyperkähler structure (see [4]) has as  $\omega_2 + i\omega_3$  the natural holomorphic symplectic structure on  $T^*(\mathbb{C}P^2)$ , and as Kähler potential for  $I$  (the natural complex structure of  $T^*(\mathbb{C}P^2)$ ) the following function:

$$(14) \quad \Omega = \log(1 + |\mathbf{z}|^2) + \sqrt{1 + 4t} - \log(1 + \sqrt{1 + 4t}),$$

with

$$(15) \quad t = (1 + |\mathbf{z}|^2)(|\mathbf{w}|^2 + \left| \sum_{j=1}^2 z^j w_j \right|^2)$$

where  $\mathbf{z} = (z^1, z^2)^T$  and  $|\mathbf{z}|^2 = |z^1|^2 + |z^2|^2$ , and similarly for  $\mathbf{w}$  and  $|\mathbf{w}|^2$ .

Consider the (local) action of  $U(1) \times U(1)$  given by

$$(16) \quad (e^{it}, e^{i\theta}).(\mathbf{z}, \mathbf{w}) = (e^{i\theta}\mathbf{z}, e^{i(t-\theta)}\mathbf{w})$$

This action preserves  $g$  and  $I$ , and its action on  $\omega_2 + i\omega_3$  simply multiplies it by  $e^{it}$ .

Restricting further to the open subset given by  $z^1 \neq 0$  and  $z^2 \neq 0$ , we make use of the following coordinate substitutions

$$\begin{aligned} z^1 &= e^{i(q^1 - q^2)} \\ z^2 &= e^{i(q^1 + q^2)} \\ w_1 &= \frac{i}{2}(p_2 - p_1)e^{i(q^2 - q^1)} \\ w_2 &= -\frac{i}{2}(p_1 + p_2)e^{-i(q^1 + q^2)} \end{aligned}$$

One can check that in the new coordinates  $(q^1, q^2, p_1, p_2)$ , which are also Darboux for  $\omega_2 + i\omega_3$ , the action is of the “standard” form

$$L_{(e^{it}, e^{i\theta})}(q^1, q^2, p_1, p_2)^T = (q^1 + \theta, q^2, e^{it}p_1, e^{it}p_2)^T$$

Using then the coordinates  $(u, v, q, r, \theta, \zeta)$  given by (1), we get that

$$\begin{aligned} |\mathbf{z}|^2 &= e^{-v}(e^{i(\bar{q}-q)} + e^{-i(\bar{q}-q)}) \\ |\mathbf{w}|^2 &= \frac{1}{4}e^{\rho+v}(|1 - \zeta|^2 e^{-i(\bar{q}-q)} + |1 + \zeta|^2 e^{i(\bar{q}-q)}) \\ \left| \sum_{j=1}^2 z^j w_j \right|^2 &= e^{\rho} \end{aligned}$$

We note that, in the new coordinates  $(u, v, q, r, \theta, \zeta)$ ,  $\Omega$  is independent of  $u$  and  $\theta$ , and the action is in standard form, so our theorem applies, and we have that, as a function of  $(v, \rho, q, \zeta)$ , with

$$\rho = \log(r^2)$$

the function  $\Omega$  becomes a function  $H$  which satisfies the 6 equations (3)-(8).

### 3. CONCLUSION

While the equations we get (equations (3)-(8)) are difficult to solve, the fact that they can be neatly written down using the Poisson bracket (2) is quite interesting, in the author’s opinion.

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